

Consequences of current conservation in systems with partial magnetic order

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Abstract

We discuss the consequences of spin current conservation in systems with $SU(2)$ spin symmetry that is spontaneously broken by partial magnetic order, using a momentum-space approach. The long-distance interaction is mediated by Goldstone magnons, whose interaction is expressed in terms of the electron Green's functions. There is also a Higgs mode, whose excitation energy can be calculated. The case of fast magnons obeying linear dispersion relation in three spatial dimensions admits nonperturbative treatment using the Gribov equation, and the solution exhibits singular behaviour which has an interpretation as a tower of spin-1 electronic excitations. This occurs near the Mott insulator state. The electrons are more free in the case of slow magnons, where the perturbative corrections are less singular at the thresholds. We then turn our attention to the problem of high- T_C superconductivity, through the discussion of the stability of the antiferromagnetic ground state in two spatial dimensions. We argue that this is caused by an effective mixing of the Goldstone ϕ and Higgs h modes, which in turn is caused by an effective ϕ condensation. The instability of the antiferromagnetic system is analyzed by studying the non-perturbative behaviour of the Higgs boson self-energy using the Dyson–Schwinger equations.

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I. INTRODUCTION

A. General picture and framework

In this paper, we discuss a momentum-space, Green's function, formulation of partial magnetic ordering. Our study is motivated phenomenologically by the problem of high- T_C superconductivity¹, where the superconducting phase appears in between anti-ferromagnetism (AF) and the metallic phase.

We ordinarily address magnetism (other than through phenomenological models) in terms of local models (e.g., the Heisenberg model) or Hamiltonians (e.g., the Hubbard model), which start from the picture that the electrons are almost localized at individual lattice sites. Their interaction, e.g., the U term in the Hubbard model, is written under this assumption.

One fundamental problem which comes with this conventional approach is that when the magnetic order is weak and the electrons are relatively free, the assumption of locality is no longer viable. Specifically, the U term of the Hubbard model is written in a spin-dependent form in accord with the Pauli exclusion principle imposed on local wavefunctions. This is no longer the case when the wavefunctions are non-local.

One would then ask, how would one formulate the origin of magnetic order, which is something that arises at the spatial scale of neighbouring lattice sites, in terms of something as delocalized as Bloch waves. The answer is that we don't. Instead of having a full description of the short-range effects, we parametrize them in terms of the energy difference ΔE between the spin states, and one magnetic condensate:

$$\langle \psi^* \psi \rangle_M \equiv \langle \psi_g^* \psi_g \rangle - \langle \psi_*^* \psi_* \rangle, \quad (1)$$

i.e., the density of occupation of electrons with magnetically favourable spin, minus the density of occupation of electrons with magnetically unfavourable spin. $\langle \psi^* \psi \rangle_M$ is obviously a function of ΔE , and is, at least in principle, calculable for a given dispersion relation and total electron density. In later discussions, we shall omit the subscript g (which stands for 'ground') and retain the subscript $*$ (which stands for 'excited'). Thus

$$\langle \psi^* \psi \rangle_M \equiv \rho - \rho_* \equiv 2S/a^d. \quad (2)$$

S is the average spin per unit cell and a^d is the volume of a unit cell. For AF, S should be

understood as referring to staggered spin.

The interaction that is responsible for magnetic order is a short-distance and short-time effect, and is generally irrelevant when discussing long range effects. All that we require to know is how the condensate enters into the quantities that are relevant to the description of long-distance effects, in particular, the Goldstone bosons which correspond physically to the magnons².

The method by which we shall discuss the Goldstone boson interaction is through the electron Green's functions since, on one hand, ρ and ρ_* are related to the integrals of the electron Green's functions and, on the other hand, the Green's functions are constrained^{3,4} by Ward–Takahashi-like identities (which, in the following, shall loosely be called the Ward–Takahashi identities) through the requirement of spin current conservation.

There is also a Higgs mode, because the high-energy (high-temperature) restoration of spin symmetry implies the presence of a mode which becomes degenerate with the Goldstone bosons in the limit of zero magnetic order.

B. The case of fast magnons in three spatial dimensions

The case of fast (large spin-wave velocity u) magnons with linear dispersion relation (i.e., like anti-ferromagnetism) in three spatial dimensions is of special interest as a model case to illustrate our approach.

When these conditions hold, we can write down a coupled system of Gribov equations. A Gribov equation^{3,5,6} is a non-perturbative self-consistency equation for the electron Green's function, which sums the leading overlapping divergences (which plague perturbation theory) to all orders.

We then solve the Gribov equations. The solution exhibits singular behaviour of the form $(E - \varepsilon)^{-3/2, 1/2}$. We interpret this as an indication of the bosonic nature of the electronic excitations. There is, furthermore, an infinite tower, with equal separation, of such excitations. This is a hint of the presence of a geometrical string (as opposed to the string-theory string which is a dynamical object with internal vibrational excitations) of variable lengths.

We do not know how to handle the more general case of magnons with arbitrary velocity, but the perturbative correction to the electron Green's function is not singular at the thresholds in the limiting case of infinitesimally small spin-wave velocity. Thus there is some

transition from correlated to uncorrelated behaviour, depending on the value of u/v_e , where u is the magnon velocity and v_e is some typical value ($\approx v_F$) of the electron velocity. The limit of fast magnons corresponds to the case where the vacuum responds collectively, and the electronic degrees of freedom is more aptly described as being due to string-like objects, whereas the limit of slow magnons corresponds to the case where the vacuum does not respond fast enough to cause significant alteration to the behaviour of individual electrons.

C. On high- T_C superconductivity

Let us turn our attention to the case of two spatial dimensions, still with linear dispersion relation for the magnons.

Since the infrared divergence is now stronger than in the case of three spatial dimensions, we expect the long-distance magnetic correlation effect to be quite radical and dominant when u is large. Superconductivity, in the form of electron pair condensation, thus requires u to be small. We demonstrate that this is usually the case, unless magnetization S is large, v_e is small and the carrier density $g(\mu)$ is small, i.e., the system is near the Mott insulator state.

However, the simple exchange of magnons between electrons is unlikely to lead to pair formation. Specifically, magnon exchange is attractive between electrons of opposite spin. Thus spin singlet pairs can form, if the pair binding energy is greater than the energy that is required to flip the spin of one of the electrons. This spin flip goes against the magnetic order which is responsible for the appearance of the Goldstone magnons which mediates the attractive interaction, and hence pair formation is unlikely. In short, we expect that the pair binding energy never exceeds the spin flip energy.

On the other hand, this restriction may be lifted for the case of an electron which is borrowed from a neighbouring site. As an illustration, the excitation of an e_* electron with an up-type spin, let us say, ostensibly costs energy, but one may borrow an e electron with an up-type spin from a site where up-type spin is favourable, i.e., a site belonging to a different sub-lattice, at a lower cost in energy.

Upon some reflection, we notice that such an effect is an instance of Goldstone-boson condensation. This then causes an effective Higgs–Goldstone interaction which, in turn, makes the AF ground state unstable against the formation of a superconducting condensate.

In view of this, we propose that the analysis should proceed through the computation of the Higgs boson Green's function to all orders. The instability should then appear in the form of an imaginary part of the Higgs boson Green's function.

The behaviour of the Higgs boson Green's function is then analyzed by summing the finite short-distance contribution to the self-energy to all orders using a Dyson–Schwinger formalism. This leads to an ordinary differential equation governing the behaviour of the self-energy as a function of the relevant Higgs–Goldstone three-point coupling. We solve this equation using a power expansion and Borel summation.

Some words of caution are necessary here.

First, an imaginary part of a Green's function is ordinarily interpreted as leading to the decay of the particle. This is not the case here since there are no final state particles to which the Higgs boson can decay. The Higgs boson ostensibly decays into a collective state of Goldstone–Higgs mixture, or, the vacuum itself decays.

Second, the instability of the vacuum does not guarantee that the stable new vacuum is superconducting. This statement is quite true in general, but we find that the expression for the imaginary part of the Green's function turns out to be of the following BCS-like^{7,8} form:

$$\Delta_{\text{SC}} \sim \omega_{\text{cut}} e^{-1/\mathcal{V}\mathcal{N}(\mu)}. \quad (3)$$

\mathcal{V} is the 4-point interaction strength. $\mathcal{N}(\mu)$ is a suitable density of states. Such an exponential form is expected in general in the case of pair-wise condensation, and hence we believe, even though this statement may not be rigorous, that the ground state is superconducting.

Third, it may appear strange that the formation of superconducting pairs (which have charge $-2e$) may be described by an imaginary part of the Higgs boson Green's function, when the Higgs boson is an object without electric charge. The point here is that the Goldstone–Higgs mixing represents the movement of electrons between the sub-lattices, and hence what we are summing to all orders is actually the behaviour of the electronic condensate. In other words, a Higgs boson is not an exciton-like electron–hole object but the fluctuation in the number density of electrons.

Fourth and last, when we discuss short-distance physics, we should in general be wary of the effect of short-distance interaction which, as we discussed earlier, are hidden and neglected. However, this applies primarily to the electrons, and not to the Goldstone and Higgs bosons which are collective excitations. Furthermore, the short-distance contributions

which we are summing to all orders are not necessarily as short as the neighbouring lattice sites.

The calculated mean-magnetization dependence of the energy gap is consistent with the observed doping dependence of the superconducting critical temperature T_C in high- T_C compounds. Our analysis suggests the co-existence of partial AF order with superconductivity in the under-doped region.

D. Organization of the paper

This paper is organized as follows.

We present a general discussion of spin-current conservation and the bosonic modes in systems with partial magnetic order in sec. II.

We derive and solve the Gribov equation for electrons interacting with fast Goldstone magnons in three spatial dimensions in sec. III.

We discuss the case of two spatial dimensions and high- T_C superconductivity in sec. IV.

The conclusions are stated at the end.

II. GENERAL ANALYSIS OF SPIN CURRENT CONSERVATION

In this section, we adapt Gribov's analysis of axial current conservation^{3,4} to the context of spin current conservation in systems with partial magnetic order.

Consider the fermionic (quasi-)electron field which is an $SU(2)_{\text{spin}}$ doublet

$$\Psi = \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow} \end{pmatrix}. \quad (4)$$

Let us define the four-vector currents. For $\hat{\Gamma}^{\mu}$ defined by the conserved electron current

$$J_{\text{electron}}^{\mu} = \Psi^{\dagger} \hat{\Gamma}^{\mu} \Psi, \quad (5)$$

we define 4-vector spin currents J_i^{μ} ($\mu = 0 \dots 3$, $i = 1, 2, 3$) of the form

$$J_i^{\mu} = \Psi_1^{\dagger} \sigma_i \hat{\Gamma}^{\mu} \Psi_2. \quad (6)$$

For example, J_3^{μ} is the up-spin current minus down-spin current. $\hat{\Gamma}^{\mu}$ can be an operator, e.g., $\hat{\Gamma}^{\mu} = i\partial/\partial x_{\mu}$.

Let us now move to the momentum space. The current vertex Γ^μ is the momentum-space counterpart of $\widehat{\Gamma}^\mu$ in the real space. In the spin-symmetric phase, the spin currents are conserved because of the conservation of electrons. That is, the following Ward–Takahashi identity is satisfied:

$$\Gamma^\mu(q_1 - q_2)_\mu \propto G^{-1}(q_1, \lambda_1) - G^{-1}(q_2, \lambda_2). \quad (7)$$

q are $d+1$ -vectors with components (q_0, \mathbf{q}) . q_0 is the energy. \mathbf{q} is either the spatial momentum or the wave number (we shall be sloppy later on for the sake of the brevity of notation). $\lambda_{1,2}$ refer to the spin states, but these are dummy indices here in the sense that $G^{-1}(q, \lambda)$ is independent of λ .

The Ward–Takahashi identity is violated in the symmetry-broken phase, since there is now an energy difference between the different spin states.

We can set locally, and without loss of generality,

$$\Psi = \begin{pmatrix} \psi_* \\ \psi \end{pmatrix}. \quad (8)$$

ψ is the spin ground state and ψ_* is the excited state. Even in the case of AF, we can still denote the spin ground state and the excited state as ψ and ψ_* , respectively.

Let us define the energy difference ΔE between the two spin states by

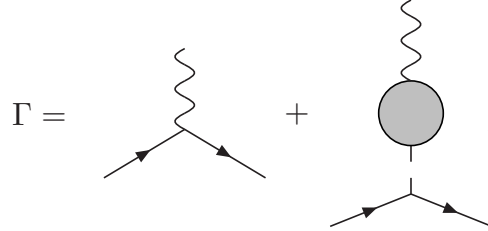
$$\Delta E = G^{-1}(q) - G_*^{-1}(q). \quad (9)$$

If G^{-1} and G_*^{-1} are both linear in energy, ΔE is given by $\epsilon_* - \epsilon$ and is constant up to a possible dependence on the spatial momentum \mathbf{q} . ΔE depends on the energy q_0 in principle. In particular, at the threshold, G^{-1} would, in general, have a singular structure corresponding to the emission and absorption of the Goldstone boson ϕ through the process $e_* \rightarrow e\phi$. We shall encounter a case in sec. III where the electron Green’s function is strongly singular in the threshold regions. The electrons move in a correlated fashion, and ΔE cannot be treated as a constant under such circumstances. However, $\Delta E \approx \text{const.}$ is a reasonable approximation when the correlation is weak and the decay width for $e_* \rightarrow e\phi$ is relatively small.

After the symmetry violation, the currents $J_{1,2}^\mu$ are no longer conserved, and the Ward–Takahashi identity is violated by

$$\Gamma^\mu(q_1 - q_2)_\mu \propto G^{-1}(q_1, \lambda_1) - G^{-1}(q_2, \lambda_2) \pm \Delta E \quad (\lambda_1 \neq \lambda_2). \quad (10)$$

This ΔE contribution is of the same form as the coupling of the Goldstone boson ϕ , and current conservation is restored by including the contribution of the Goldstone boson. This is the case even when ΔE is not constant. Specifically, spin current conservation is restored for the vertex modified by the inclusion of the Goldstone boson,



$$\Gamma = \text{[diagram 1]} + \text{[diagram 2]} . \quad (11)$$

The wavy lines indicate the spin current, and the dashed line indicates the Goldstone boson. The two-point function in the second term is nothing but an electron loop. There is nothing strange in this result, since the Goldstone boson arose in the first place as the longitudinal component of the spin current. After taking away the longitudinal component, the remaining part is transverse and therefore satisfies the Ward–Takahashi identity.

The Goldstone bosons ϕ_1 and ϕ_2 correspond to the SU(2) rotation,

$$\mathcal{U}(\phi_1, \phi_2) = \exp \left[i f^{-1} \sum_{i=1}^2 \phi_i \sigma_i \right] , \quad (12)$$

and they correspond physically to the magnons. f is the Goldstone boson form factor which, by virtue of eqn. (11), is calculated as the strength of the current–Goldstone-boson two-point amplitude. Their coupling strength with fermions is then given by $f^{-1}\Delta E$. The Higgs boson mass M_h , or excitation energy, is evaluated in a similar fashion⁴ to the evaluation of f .

A. Case of quadratic dispersion relation

For ferromagnetism, the spin-wave velocity is known to have a quadratic dispersion relation, $\omega \propto \mathbf{k}^2$, for small \mathbf{k} . That is, the Green’s function is given by $D_\phi^{-1} = E - c\mathbf{k}^2$, where c is some constant. In this case, evaluating the electron loop in the two-point function shown in the second term of eqn. (11), between the spin current and the Goldstone boson, should yield $f \times (1, c\mathbf{q})$ in the soft limit such that eqn. (11) satisfies the Ward–Takahashi identity. The form factor f has the dimension of $(\text{length})^{-d/2}$.

By considering the 0th component of the two-point function in the soft limit, we obtain

$$f^2 = -2 \times \int \frac{d^{d+1}k}{(2\pi)^{d+1}i} G(k) G_*(k) \Delta E(k). \quad (13)$$

The factor 2 is from contracting the Pauli matrices, or for the two spin orientations. The corrections to the electron Green's function are included in this expression. As for the vertex corrections, we are taking the current vertex to be bare and take the value 1 in its 0th component, whereas the Goldstone-boson vertex is renormalized (as otherwise there will be double counting). The latter correction gives rise to the renormalization of ΔE . f is just a number. This expression, and the results that follow, are therefore exact.

By eqn. (9), and because the integral of a Green's function is the number density, we find

$$f^2 = 2(\rho - \rho_*). \quad (14)$$

ρ is the number per unit volume or area.

We can define the mean magnetization S , $0 \leq S \leq 1/2$, by

$$2S/a^d = \rho - \rho_*, \quad (15)$$

to obtain

$$f^2 = 4S/a^d. \quad (16)$$

Let us temporarily consider a magnetic order-parameter field Φ , whose rotational invariances correspond to the Goldstone bosons. It is easy to see that $f = 2v$, where v is the vacuum expectation value of Φ . This implies $v^2 = S/a^d$, which seems natural.

B. Case of linear dispersion relation

In the case of a linear dispersion relation for the Goldstone boson as in the AF magnons, let us proceed analogously to the above case of quadratic dispersion relation. Now the Green's function of the Goldstone boson is given by $D_\phi^{-1} = E^2 - \hbar^2 u^2 \mathbf{k}^2$. We then see that the two-point function shown in the second term of eqn. (11), between the spin current and the Goldstone boson, is now given by $f q_\mu$ (with the spin-wave velocity $u = 1$), where q_μ is the 4-momentum. In particular, the two-point function vanishes in the soft limit. In the case of AF, this is due to the cancellation between the two spin orientations, or between the two sub-lattices.

The form factor f^{-1} has the dimension of $(\text{energy} \times \text{length}^d)^{1/2}$. Let us make use of

$$\text{wavy line} \text{---} \text{circle} \text{---} \text{wavy line} = -f q_\nu \text{wavy line} \text{---} \text{circle} \text{---} \text{wavy line}. \quad (17)$$

Note that we have been rather loose in the notation here: this equation is valid only in the sense that the cancellation between the sub-lattices has been taken into account on the left-hand side. If not, we would be equating something that is finite in the soft limit on the left-hand side against something which vanishes on the right-hand side.

It follows that, for small momenta,

$$f^2 g_{\mu\nu} = - \text{wavy line} \text{---} \text{circle} \text{---} \text{wavy line}. \quad (18)$$

For the 0-0 component of this quantity, and for nearly constant ΔE , we obtain

$$f^2 = 2 \times \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1}{\Delta E}. \quad (19)$$

The factor 2 is again from contracting Pauli matrices. The region of integration covers all points where one spin state is occupied and the other spin state is vacant.

Let us define the mean magnetization S , $0 \leq S \leq 1/2$, again by

$$2S/a^d = \rho - \rho_*, \quad (20)$$

We then have

$$f^2 = \frac{4S}{a^d \Delta E}. \quad (21)$$

In terms of the order-parameter description, $f = 2v$ as before, and this expression seems natural from this perspective.

C. Relationships among ΔE , $\langle \psi^* \psi \rangle_M$, S and u

$\langle \psi^* \psi \rangle_M$ is controlled by ΔE when ΔE is nearly constant. If we denote the density of states (of both spin states) by $g(E)$ and if this density of states can be taken to be roughly constant near the chemical potential μ , we see immediately by comparing the occupation of the two spin states that

$$\rho - \rho_* = g(\mu) \Delta E / 2, \quad (22)$$

or

$$\Delta E = \frac{4S}{g(\mu)a^d}. \quad (23)$$

Hence

$$f^2 = \begin{cases} g(\mu)\Delta E & \text{(quadratic dispersion relation),} \\ g(\mu) & \text{(linear dispersion relation).} \end{cases} \quad (24)$$

In secs. II A and II B, we calculated f^2 through the consideration of the time component of the spin-current-magnon two-point function. Looking at the space component of the same amplitude should give us the spin-wave velocity, in principle. However, the calculation is messy, and it can depend on the threshold behaviour of G , G_* and ΔE .

A simpler and more intuitive way to obtain the spin-wave velocity, for a given form of the dispersion relation, is to equate the excitation energy $\hbar\omega_\phi$ of the π/a magnon mode with the excitation energy of e to e_* at π/a , because these are equivalent. This tells us, quite generally, that

$$\hbar\omega_\phi(\pi/a, 0, 0) = \varepsilon_*(\pi/a, 0, 0) - \varepsilon(\pi/a, 0, 0) = \Delta E. \quad (25)$$

In particular, for a linear dispersion relation with velocity u , we obtain

$$\frac{\hbar u \pi}{a} = \Delta E \quad \Rightarrow \quad u = \frac{a \Delta E}{\hbar \pi} = \frac{4S}{\hbar \pi g(\mu) a^{d-1}}. \quad (26)$$

Note that we cannot expect the dispersion relation to be exactly linear up to π/a . This is an approximation.

D. The Higgs boson

As we expect spin symmetry to be restored in the high energy (temperature) limit, there is necessarily a Higgs boson h to complement the Goldstone bosons and provide the remaining local rotational degree of freedom. By this symmetry, the Higgs boson couples to fermions with the same coupling strength ($f^{-1}\Delta E$) as the Goldstone bosons, but there is now an energy gap. This energy gap is due to corrections below the high energy scale. Explicitly, the corrections are given by the self-energy integral:

$$\Sigma_h = - \int \frac{d^{d+1}k}{(2\pi)^{d+1}_i} (f^{-1}\Delta E)^2 [G(k)G(k-q) + G_*(k)G_*(k-q)]. \quad (27)$$

We then compare this with the corresponding integral for the Goldstone bosons:

$$\Sigma_\phi = -2 \int \frac{d^{d+1}k}{(2\pi)^{d+1}_i} (f^{-1}\Delta E)^2 G(k)G_*(k-q). \quad (28)$$

We see that

$$\Sigma_h(q + (\Delta E, \mathbf{0})) = \Sigma_\phi(q), \quad (29)$$

i.e., the correction to the Higgs boson at energy $\Delta E + E$ is the same as that to the Goldstone boson at energy E . Since the Goldstone bosons have zero excitation energy, the excitation energy for the Higgs boson is thus equal to ΔE (at least for almost constant ΔE). It follows that the Green's functions for the Higgs boson are given by

$$D_h^{-1}(q) = D_\phi^{-1}(q) - \begin{cases} \Delta E & \text{(quadratic dispersion relation),} \\ (\Delta E)^2 & \text{(linear dispersion relation).} \end{cases} \quad (30)$$

The above derivation is somewhat intuitive. Let us derive the same results more rigorously, using a UV counterterm argument. The excitation energy is ordinarily found by the condition that the radiative correction vanishes, i.e., $\Sigma = 0$, when the excitation is on the mass shell. This Σ should be the sum of Σ_h and Σ_{UV} , where Σ_{UV} is the high-energy contribution due to $\langle \psi^* \psi \rangle_M$, and is the same between ϕ and h . The condition that $\Sigma_\phi + \Sigma_{UV} = 0$ at $q = 0$ completely fixes Σ_{UV} as

$$\Sigma_{UV} = +2 \int \frac{d^{d+1}k}{(2\pi)^{d+1}i} (f^{-1} \Delta E)^2 G(k) G_*(k) = -2 f^{-2} \Delta E \langle \psi^* \psi \rangle_M. \quad (31)$$

From the requirement $\Sigma_h + \Sigma_{UV} = 0$, we then obtain the excitation energy ΔE . That there is a solution to $\Sigma = 0$ is a sufficient proof for the existence of the Higgs boson.

III. THE NON-RELATIVISTIC GRIBOV EQUATION

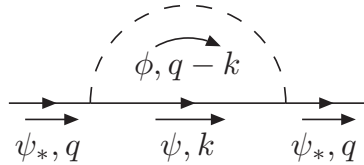


FIG. 1. The electron self-energy diagram.

The exchange of Goldstone bosons is the leading long-distance contribution to the electron self-energy. Corresponding to fig. 1, the self-energy is given by

$$\Sigma_*(q) = 2 \int \frac{d^{d+1}k}{(2\pi)^{d+1}i} f^{-2} \Delta E^2 G(k) D(q - k), \quad (32)$$

and similarly for $\Sigma(q)$. The factor 2 sums over the two Goldstone modes. D is the propagator of the Goldstone bosons. This expression contains all corrections other than the vertex correction which leads to a correction to ΔE .

In general, $G(k)$ and $G_*(k)$ become singular in the threshold regions where the decays $e_* \rightarrow e\phi$ and $e \rightarrow e_*\phi$ (the latter being to a virtual e_*) open up. This is fine at the leading order, but since corrections to $G(k)$ and $G_*(k)$ affect ΔE , the general expression becomes troublesome, and we do not know how to handle it. This problem of ‘overlapping divergences’ is, of course, a well-known generic problem of perturbation theory.

However, we have one exception in the case of three spatial dimensions, with a linear dispersion relation of the Goldstone boson, and with large u . We may, in this case, employ the Gribov equation framework^{3,5,6}. Let us therefore specialize to this situation here.

A. Derivation of the Gribov equation

To obtain the Gribov equation, we apply

$$\frac{\partial^2}{\partial q_0^2} - \frac{1}{u^2} \frac{\partial^2}{\partial \mathbf{q}^2}, \quad (33)$$

to eqn. (32). The integral sign then disappears because of the following identity:

$$\left[\frac{\partial^2}{\partial q_0^2} - \frac{1}{u^2} \frac{\partial^2}{\partial \mathbf{q}^2} \right] \frac{1}{(q_0 - k_0)^2 - u^2(\mathbf{q} - \mathbf{k})^2 + i\epsilon} = \frac{4\pi^2 i}{u^3} \delta^{(4)}(q - k). \quad (34)$$

In other words, $D(q)$ is a Green’s function of the 4-dimensional Laplace equation in the energy–momentum space. We have assumed that the boundaries in \mathbf{k} do not affect the argument, because u is large. That is, only small \mathbf{k} is exchanged, typically. We then obtain

$$\left(\frac{\partial^2}{\partial q_0^2} - \frac{1}{u^2} \frac{\partial^2}{\partial \mathbf{q}^2} \right) \Sigma_*(q) = \frac{(f^{-1}\Delta E)^2}{2\pi^2 \hbar^3 u^3} G(q). \quad (35)$$

So far, this is just an algebraic manipulation which is valid at the one-loop order, but we now notice that something else has occurred to the right-hand side. To see this, let us consider the general N -loop perturbative contribution $\delta\Sigma$ to the self-energy. The leading contribution is from regions with overlapping divergences, where there may be a number of large logarithms being involved, in the form, for example,

$$\delta\Sigma \sim \log(r_1) \log(r_2), \quad (36)$$

where r_1 and r_2 are some large ratios. Now we consider differentiating this expression twice. If we differentiate $\log(r_1)$ once and $\log(r_2)$ once, we obtain something like $1/r_1 r_2$, and both of the large logarithms have disappeared. On the other hand, if we differentiate only $\log(r_1)$, the other logarithms remain intact, and so this is the leading contribution to the double derivative of $\delta\Sigma$.

Since these large logarithms are due to the integrals of $D(q)$, it follows that the double derivative, when applied to the general N -loop expression for self-energy, removes one or the other of the Goldstone boson propagators, and, the final contribution is of the form given in eqn. (35). The higher loop contributions amount to the vertex corrections, and so eqn. (35) is in fact valid to all orders in the sense of summing the leading overlapping divergences.

Note that this approach is quite distinct from methods such as the leading-logarithm summation, which do not deal with the problem of overlapping divergences. More detailed discussion of the various contributions to the Gribov equation is available in the papers³ by Gribov.

Equation (35) can be turned into a self-consistency equation for the electron Green's function in the limit of large u , since the left-hand side then becomes equivalent to the energy double derivative of $-G_*^{-1}(q)$, i.e.,

$$-\frac{\partial^2}{\partial q_0^2} G_*^{-1}(q) = \frac{(f^{-1}\Delta E)^2}{2\pi^2 \hbar^3 u^3} G(q). \quad (37)$$

Note that, in general, $G^{-1} = G_0^{-1} - \Sigma$, and we are assuming that G_0^{-1} has the form $G_0^{-1} = q_0 - \varepsilon(\mathbf{q})$. The spatial-momentum dependence has dropped out of the expression, and we can write this equivalently as

$$\begin{cases} (G_*^{-1})'' = -f_R^{-2}(G^{-1} - G_*^{-1})^2 G, \\ (G^{-1})'' = -f_R^{-2}(G^{-1} - G_*^{-1})^2 G_*, \end{cases} \quad (38)$$

with $f_R^2 = 2\pi^2 \hbar^3 u^3 f^2$. Prime refers to the energy derivative.

B. Solution of the Gribov equation

Let us proceed to solve eqns. (38). We first consider the expression

$$G_*(G_*^{-1})'' - G(G^{-1})'' = 0, \quad (39)$$

which follows from eqns. (38). We then use the identity $x^{-1}x'' \equiv (\ln x)'' + ((\ln x)')^2$:

$$(\ln G_*^{-1}/G^{-1})'' + ((\ln G_*^{-1})')^2 - ((\ln G^{-1})')^2 = 0. \quad (40)$$

Let us define $z = G_*^{-1}/G^{-1}$, so that

$$(\ln z)'' + (\ln z)'(\ln G_*^{-1}G^{-1})' = 0. \quad (41)$$

This can be integrated once, and we obtain

$$\ln(\ln z)' + \ln(G_*^{-1}G^{-1}) = \text{const.}, \quad (42)$$

or,

$$(\ln z)' = CGG_*. \quad (43)$$

C is a constant of integration. This implies

$$G^2 = C^{-1}z', \quad G_*^2 = C^{-1}z'/z^2. \quad (44)$$

Let us make use of eqn. (44) to eliminate the energy derivatives in eqns. (38). This gives us, almost trivially,

$$G^3 \frac{d^2}{dz^2} G = (f_R C)^{-2} (z + z^{-1} - 2). \quad (45)$$

The equation for G_* is obtained by substituting G_* for G and z^{-1} for z .

This equation is simple, but does not seem to have a simple analytical solution. However, we obtain the following limiting behaviours:

- When G diverges, $z = G_*^{-1}/G^{-1}$ should also diverge. We see that the limiting expressions are given by $G = (f_R C)^{-1/2}(-16z^3/3)^{1/4}$ and $G_* = (f_R C)^{-1/2}(-16/3z)^{1/4}$. That is, G_* vanishes.
- When G_* diverges, G and z vanish. The limiting expressions are given by $G = (f_R C)^{-1/2}(-16z/3)^{1/4}$ and $G_* = (f_R C)^{-1/2}(-16/3z^3)^{1/4}$.

In order to obtain a numerical solution of eqn. (45), it is desirable to replace z with something that has a limited range. In this regard, we notice that the left-hand side of eqn. (45) has a conformal symmetry with respect to transformations on z . This symmetry becomes more manifest when we return to eqns. (38) and now eliminate G and G_* using eqn. (44). After some elementary algebra, we obtain

$$\frac{z'''}{2z'} - \frac{3}{4} \left(\frac{z''}{z'} \right)^2 = f_R^{-2} (z + z^{-1} - 2). \quad (46)$$

We then see that the left-hand side of this equation is invariant under the Möbius transformation:

$$z \longrightarrow w = \frac{az + b}{cz + d}. \quad (47)$$

It turns out that z is negative. Let us therefore use

$$w = \frac{1+z}{1-z}, \quad z = -\frac{1-w}{1+w} \quad (-1 \leq w \leq 1). \quad (48)$$

We also define

$$y = (16f_R^{-2}/3)^{-1/4} \sqrt{\frac{dw}{dE}} = \frac{(3f_R^2 C^2/4)^{1/4}}{G^{-1} - G_*^{-1}} \equiv \frac{(3f_R^2 C^2/4)^{1/4}}{\Delta E}. \quad (49)$$

We then obtain

$$y^3 \frac{d^2 y}{dw^2} = -\frac{3}{4(1-w^2)}. \quad (50)$$

Note the similarity with eqn. (45). We may have obtained eqn. (50) directly from eqns. (45) and (44) using eqn. (48).

Now y vanishes when $w = \pm 1$, and the limiting expression is given by $y \rightarrow (1-w^2)^{1/4}$. We should use this boundary condition to solve eqn. (50) numerically, but this is difficult. We therefore use the boundary condition $dy/dw = 0$ at $w = 0$ and substitute some numbers for y at $w = 0$ and, by trial and error, establish the value of y at $w = 0$ that leads to the correct behaviour at $w = \pm 1$. The result of this calculation is shown in fig. 2. The numbers were obtained using the classical Runge–Kutta method with the step size of 0.0005, and were found to be stable against modifications to the step size.

We found that the value of y is approximately 1.061 when $w = 0$. We see that $y = (1-w^2)^{1/4}$ and $y = 1.061(1-w^2)^{1/4}$ are both good approximations to the actual behaviour of y .

Let us define the scaled energy x as $x = (E - \varepsilon_*) \sqrt{16f_R^{-2}/3}$, so that $y, z \rightarrow 0$ as $x \rightarrow 0$. By eqn. (49), we obtain

$$\int dx = \int \frac{dw}{y^2(w)}. \quad (51)$$

The integral on the right-hand side is finite when the integration is performed between $w = \pm 1$. When $w = \pm 1$, the derivative of w with respect to x vanishes. w is therefore periodic in x . Let us, for the sake of simplicity, adopt $y = (1-w^2)^{1/4}$. This leads to

$$w = \cos x, \quad z = -\tan^2(x/2). \quad (52)$$

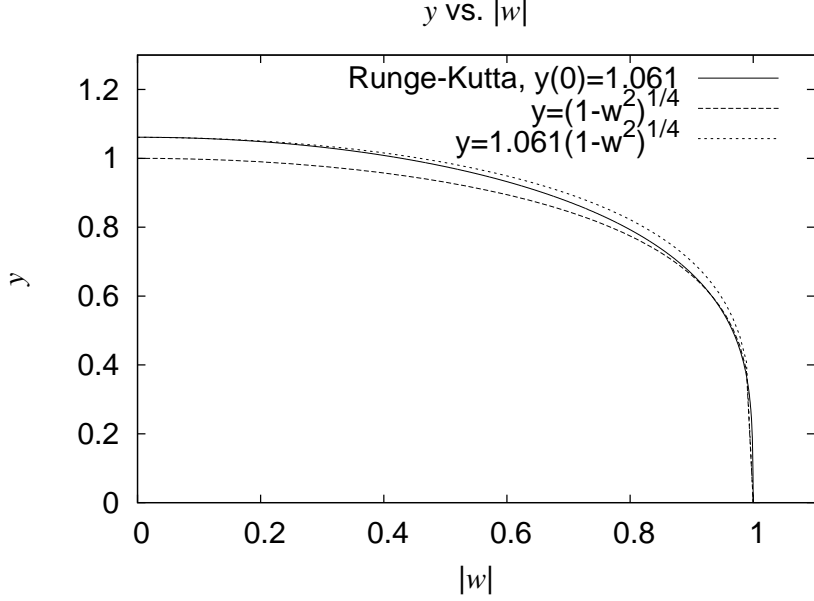


FIG. 2. y vs w . We compare the numerical results with the limiting expression $y \propto (1 - w^2)^{1/4}$.

C. Nature of the solution

We have seen in the above that the solution has the following form:

$$z \approx -\tan^2(x/2) \quad (53)$$

This implies that, by eqn. (44),

$$G \approx \pm \sqrt{-C^{-1} \tan(x/2) \sec^2(x/2)}, \quad G_* \approx \mp \sqrt{-C^{-1} \cot(x/2) \operatorname{cosec}^2(x/2)}. \quad (54)$$

As $x \rightarrow 0$, these functions behave as

$$G \rightarrow \pm \sqrt{-C^{-1}x/2}, \quad G_* \rightarrow \mp \sqrt{-8C^{-1}/x^3}. \quad (55)$$

This is clearly a rather exotic behaviour for the Green's function of an electron. For instance, the integral of the Green's function, or the electron number, becomes undefinable. Note that this does not necessarily invalidate our starting point since we can still define $\langle \psi^* \psi \rangle_M$ as a finite number. Furthermore, the non-local correlator $\langle \psi^*(t) \psi(t + \Delta t) \rangle$ is definable and is non-zero (at, in this case, discrete values of Δt). That is, the field theory is non-local.

One interpretation of this exotic power behaviour is through the Levinson theorem which states that the number of states in between energies E_1 and E_2 is given by the difference in

the phase shifts $\delta(E_1, E_2)$ under scattering:

$$N = \frac{1}{\pi} [\delta(E_2) - \delta(E_1)]. \quad (56)$$

In this sense, we can say that there is $-1/2$ electron of the e type and $+3/2$ electron of the e_* type at $x = 0$. On the other hand, when z diverges, we obtain $-1/2$ e_* electron and $+3/2$ e electron.

This result indicates that there is no longer a physical electron. However, if we sum the states that are present at $x = 0$, we see that there is $-1/2 + 3/2 = 1$ state, whose charge is $-e$ and spin is, if such a sum may be made, $|-1/2(1/2) + 3/2(-1/2)| = 1$. In this sense, the electronic excitations are bosonic.

What is the nature of these bosonic excitations?

Our solution implies that there is a tower of such states, at each phase-space point, with equal separation in energy (of approximately $f_R \pi \sqrt{3}/4$) and alternating spin direction. Note that the number of states in each excited state is the same. This implies that the excitations are in the form of one-dimensional objects. This then suggests the existence of string-like objects (such as the trajectory drawn by spin-flipped electrons), which bind together the electrons. The quantized energies correspond to the length of the string. The string is not a dynamical object with an internal degree of freedom, as otherwise we cannot explain the one-dimensional nature of the excitations.

We should add that a power-like behaviour of the Green's function is obtained also in the case of simple photon-exchange interaction. In this case, the Gribov equation is given by

$$(G^{-1})'' = \frac{\alpha}{\pi} G((G^{-1})')^2. \quad (57)$$

$\alpha \approx 1/137$ is the fine structure constant. This is in the absence of screening. It is easy to see that this has the following general solution:

$$G^{-1} \propto (E - \varepsilon)^{1/(1-\alpha/\pi)}. \quad (58)$$

According to the above argument based on the Levinson theorem, the number of electronic states goes up as a result of the repulsive Coulomb interaction. There seems to be a contribution due to plasmon-like charge oscillation states here.

D. The case of slow magnons

The above discussion is for the case of fast magnons with linear dispersion relation in three spatial dimensions. This case is special in that the leading overlapping divergences can be dealt with by using the Gribov equation framework.

Let us consider relaxing one or the other of these conditions.

First, we do not know what the situation may be with respect to changing the linear dispersion relation to a quadratic dispersion relation.

Second, the infra-red divergence is stronger in 2-d. We thus expect the long-distance correlations to be even more dominant than in the 3-d case. On the other hand, we know that there is no long-distance correlation in the 1-d case because of the strong quantum fluctuations. Our intuitive guess (which is only a wild guess) is that the string picture is valid both for 3-d and 2-d, because the string is a one-dimensional object which can be embedded in both 3-d and 2-d. The 1-d case is obviously different in this picture.

Third and last, let us consider the case of very slow magnons. This is interesting because although eqn. (32) diverges, it is almost constant with respect to variations in the external 4-momentum q . Explicitly, let us consider replacing eqn. (32) by the expression

$$\Sigma_*(q)|_{\text{small } u} \approx 2 \int \frac{d^{d+1}k}{(2\pi)^{d+1}i} f^{-2} \Delta E^2 \frac{1}{(q_0 - k_0)^2 - u^2 \kappa^2 + i\delta} \frac{1}{k_0 - \mathbf{k}^2/2m \pm i\delta} \quad (59)$$

in the limit of small u . κ is some typical spatial momentum scale. Then this expression is obviously independent of \mathbf{q} , and it can only depend mildly on q_0 . More explicitly, we find that, as we would naturally expect, Σ_* depends on q_0 almost only through $g(q_0)$:

$$\Sigma_*(q)|_{\text{small } u} \approx -\frac{g(q_0)}{u\kappa} \ln \left(\frac{q_0 - \varepsilon_h + i\delta}{q_0 - \varepsilon_l + i\delta} \right). \quad (60)$$

This particular expression uses the approximation that $g(E)$ is almost constant. $\varepsilon_{h,l}$ are the upper and lower limits of the electron dispersion relation, and are expected to be very far away from q_0 .

This finding, that Σ_* , and Σ by a similar calculation, are almost independent of q , is quite intuitive. A slow magnon field tends to stay fixed at a point in space-time, and therefore its interaction is independent of \mathbf{q} . When Σ is independent of \mathbf{q} , there is no divergence at thresholds (which depend on \mathbf{q}) and, equivalently, no singularities at thresholds in q_0 .

The divergent integral of eqn. (59) can therefore be treated as a constant to a good

approximation. This gives rise to the renormalization of ε . ΔE is unaffected, and therefore there are no other corrections to Σ at any order (consider the Dyson–Schwinger equations).

The above argument is for the limiting case of zero spin-wave velocity. This is also intuitive from considering the BCS theory, where the electron-pair interaction is governed both by the coupling constant and the cut-off at the Debye frequency ω_D which corresponds to $u\kappa$ in the above case. There is no superconductivity in the limit of zero ω_D . What, then, will be the case when $u\kappa$ is small but non-zero? We shall argue in the next section that even in that case, we cannot have electron-pair formation due to magnon exchange. Therefore electrons remain more or less free. Note that the formation of an electron-pair condensate should reflect in the electron Green’s function through its imaginary part (and possibly also the real part).

Thus, at least for the case of linear spin-wave dispersion relation, the behaviour of the electrons changes qualitatively depending on whether the spin-wave velocity is fast or slow. For fast magnons ($u \gg v_F$), the response of the (magnetic) vacuum is fast and collective, and the electrons are under its influence. For slow magnons ($u \ll v_F$), the response of the vacuum is slow and partial, and the electrons move around more freely.

These results are quite intuitive, because according to eqn. (26), u is large when S is large and $g(\mu)$ is small. If the electron dispersion relation goes as $\epsilon \sim k^2/2m$, one can see that the $u > v_F$ when v_F is small and S is large. In other words, the case of fast spin-wave velocity occurs only near the Mott insulator state or some such state with large S and small electron velocity. Spin-wave velocity is slow when S is small, i.e., when the magnetic order is broken, so that the electrons move around more freely.

IV. A STUDY OF HIGH- T_C SUPERCONDUCTIVITY

Since high- T_C cuprates are to a good approximation 2-d systems with AF interaction, let us focus on the case corresponding to these conditions.

We saw at the end of the previous section how electrons start to become more free when the magnetic order is weakened.

We then consider how superconducting pairs may be formed.

In the exchange of a magnon between two electrons, the interaction is attractive between e and e_* , but is repulsive between e and e , or e_* and e_* . This implies that for the formation

of an ee_* pair, the binding energy must exceed ΔE , as otherwise there is insufficient energy to excite e to e_* . But it would be unnatural that the interaction strength is so large as to allow such pairs to form. For instance, the BCS binding energy is expected to be less than the cut-off energy (the Debye frequency in BCS theory) which is given by ΔE . If such an interaction arises at all, it can only be due to something like overscreening, and this circumstance is more naturally associated with SDW (spin-density wave) or some such magnetic structure.

We thus believe that superconductivity through magnetic exchange occurs through a different mechanism, and this is through, let us say, the mixing between the two (e.g., spin-up and spin-down) sub-lattices that constitute the AF ground state (but note that a completely analogous argument can be made for ferromagnetism: this argument is not specific to AF).

Through this mechanism, it is possible to ‘borrow’ an e_* electron from a neighbouring site to form spin-singlet pairs without paying the excitation energy ΔE .

Let us estimate the size of the superconducting gap due to this mechanism, using the BCS expression quoted in the introduction (eqn. (3)):

$$\Delta = 2\omega_{\text{cut}} \exp(-1/\mathcal{V}\mathcal{N}(\mu)), \quad (61)$$

where \mathcal{V} is the effective 4-point interaction strength, and is estimated by f^{-2} times some function $\Theta(\theta)$ of a ‘mixing-angle’ for the ‘mixing’ between the borrowed electrons and the initial electrons. The cut-off, ω_{cut} , is estimated by ΔE . $\mathcal{N}(\mu)$ here is the number density of these ‘borrowed’ electrons. Altogether, we thus expect

$$\Delta \sim \Delta E \exp(-g(\mu)/\Theta(\theta)\mathcal{N}(\mu)). \quad (62)$$

The expression which we shall obtain later on corresponds to $\mathcal{N}(\mu) = M_h/2\pi\hbar^2$, up to a possible constant factor. This is the number density of Higgs bosons if the chemical potential were exactly at the excitation energy of a $\mathbf{k} = 0$ Higgs boson.

There is one conceptual problem with this approach, namely that this mixing between the sub-lattices is not an electronic excitation but essentially an alteration of the AF ground state, in the sense that the flipping of the spin due to this effect is associated with the local modification of the vacuum from, say, $\Phi = (0, v)$ to $\Phi = (v, 0)$, where Φ is an order parameter. In other words, the phenomenon that we are describing here is, we believe,

Goldstone-boson condensation. What we need to discuss then is the stability of the AF ground state after Goldstone-boson condensation has taken place. This instability appears via the Higgs–Goldstone coupling which is due to the mixing of the Goldstone boson with the Higgs boson. This mixing is, as we shall see, a direct consequence of the Goldstone-boson condensation.

A physical picture for this effect is as follows. In the AF state, moving an electron to its neighbouring site, without disturbing the AF order, requires flipping the spin of the electron. This takes place via the emission of a magnon. However, moving the electron without raising it to the spin-excited state corresponds to the Higgs boson, and so the Higgs boson couples to the Goldstone boson, or there is effectively a mixing between the Higgs boson and the magnon.

We know how one might go about calculating this effect, but the full description is cumbersome, and we shall parametrize it by an effective mixing angle θ . This allows us to calculate the corrections to the Higgs boson Green’s function.

A. Higgs–Goldstone coupling

In our picture, Goldstone-boson condensation by itself costs energy, which is compensated by the superconducting condensation energy. ϕ condensation can at least formally be parametrized by a negative ϕ mass squared $-\mu_\phi$. This leads to the mass matrix:

$$M_{\phi,h}^2 u^4 = \begin{pmatrix} -\mu_\phi & 0 \\ 0 & M_h^2 u^4 - \mu_\phi \end{pmatrix}. \quad (63)$$

But a Goldstone boson is necessarily massless, and so the physical modes are obtained by rotating ϕ and h by a mixing angle $\theta_{h\phi}$ such that ϕ becomes massless. Some off-diagonal counter-terms are obviously necessary to stabilize the vacuum in this way, but let us skip the details here.

Let us denote this angle $\theta_{h\phi}$ simply as θ in the following. θ is determined by the requirement that the total energy is minimized. That is, if superconductivity occurs, θ is such that the condensation energy plus the extra energy due to exciting the Goldstone bosons is minimized. This is clearly too complicated, and so let us treat θ as a free parameter in the following.

We denote the mixed states h' and ϕ' as

$$\begin{pmatrix} h' \\ i\phi' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} h \\ i\phi \end{pmatrix}. \quad (64)$$

In general, only one linear combination of Goldstone bosons is involved, and the other Goldstone boson mode remains orthogonal to both h' and ϕ' .

In terms of μ_ϕ introduced above, we have

$$\tan \theta = \sqrt{\frac{\mu_\phi}{M_h^2 u^4 - \mu_\phi}} \approx \sqrt{\mu_\phi}/M_h u^2. \quad (65)$$

Let us assume that μ is small and the mixing angle is not huge, so that $M_{h'} u^2$ can be approximated by $M_h u^2 = \Delta E$ as calculated in sec. II D. In principle, θ can be measured by measuring the discrepancy between electronic excitation energy ΔE and the Higgs-boson excitation energy, but this is probably too delicate to be useful.

In terms of h and ϕ , the $h\phi\phi$ coupling is given by

$$\frac{\hbar^2}{4} f h (f^{-1} \partial_\mu \phi_2)^2. \quad (66)$$

We derived this from the kinetic-energy part of the effective Lagrangian density for the order parameter field,

$$\mathcal{L}_{\text{kinetic}} = \frac{\hbar^2}{2} (\partial_\mu \Phi_i)^2 = \frac{\hbar^2}{2} \left[\left(\frac{\partial \Phi_i}{\partial t} \right)^2 - u^2 (\nabla \Phi_i)^2 \right], \quad (67)$$

using $v = f/2$ as before. Since the kinetic-energy part of the effective Lagrangian is largely model independent, we believe that this result is fairly general.

We are interested in the $\phi' h' h'$ coupling. From eqn. (66), for near on-shell Higgs bosons, and in terms of h' and ϕ' , the three-point coupling Γ is found to be given by

$$\frac{\Gamma}{2!} \approx \frac{\hbar^2 f^{-1} (M_{h'} u^2)^2}{4} \sin \theta (1 - 3 \cos^2 \theta). \quad (68)$$

The $2!$ is the identical particle factor. Note that this Γ is different from the electron-magnon vertex which appeared in sec. II. Even though there is, in principle, a four-point function in addition to the three-point interaction, we do not know how to sum both three-point and four-point vertices to all orders, and therefore we neglect it here.

One simple estimate for θ is obtained by saying that Γ^2 is maximized. We find that Γ^2 is maximized when $\theta = \pi/2$, which is too large to be acceptable. A local maximum occurs

for $\sin \theta = \sqrt{2}/3$, when θ is 28 degrees and $\sin^2 \theta (1 - 3 \cos^2 \theta)^2 = 32/81$. This seems more appropriate, but let us not adopt any specific value here.

For the sake of later analysis, let us introduce the dimensionless coupling x , defined by

$$x = \frac{-\Gamma^2}{4\pi M_h^3 u^6} \times \frac{1}{(\hbar u)^2}. \quad (69)$$

Note again that this x is different from the normalized energy which appeared in sec. III B. This is then given by

$$x \approx -\frac{f^{-2} M_h u^2}{16\pi (\hbar u)^2} \sin^2 \theta (1 - 3 \cos^2 \theta)^2 = -\frac{\Delta E}{16\pi (\hbar u)^2 g(\mu)} \sin^2 \theta (1 - 3 \cos^2 \theta)^2. \quad (70)$$

We found in sec. II C that $\Delta E = S/g(\mu)a^2$ and that $\hbar u \pi/a = \Delta E$. We can use these relations to eliminate ΔE and $g(\mu)$:

$$x \approx -\frac{\pi}{16S} \sin^2 \theta (1 - 3 \cos^2 \theta)^2 = -\mathcal{O}(10^{-2} \sim 10^{-1}) \frac{1}{S}. \quad (71)$$

This is somewhat surprising in the sense that it implies that the dimensionless coupling x grows as the magnetization falls. However, this behaviour is correct, as we shall show later, since this leads to the survival of magnetic order in the large-magnetization region.

B. One-loop results

Our objective is to analyze vacuum instability due to the exchange of Goldstone magnons. In two spatial dimensions and in the leading-order approximation, the potential due to the single exchange of a magnon is proportional to $\log(r)$ in the real space. The potential is therefore unbound as r goes to infinity. This makes Higgs bosons long-distance, or infrared, confined. This affects the way that the Higgs bosons can be excited. Presumably this disfavours single Higgs-boson excitations.

This confinement only affects the excitations and does not affect the structure of the vacuum. Hence it is quite distinct from the superconductivity phenomenon which we are interested in, which is something that actually alters the structure of the vacuum. In short, we are interested in Higgs super-criticality, which occurs when the binding energy exceeds the energy that is needed to create a Higgs boson, and is an $\mathcal{O}(\hbar/M_h u) = \mathcal{O}(a)$ short-distance effect. We should add that the super-conducting coherence length is indeed relatively short in the real high- T_C systems¹, which adds support to our discussion.

The finite ultra-violet contributions can be summed to all orders. Let us start with the one-loop self-energy correction to the Higgs boson propagator. The corresponding Feynman diagram is shown in fig. 3. All momenta are 2 + 1-vectors, i.e., with two space and one time components.

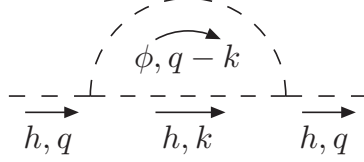


FIG. 3. The one-loop self-energy diagram.

In the following, let us adopt the convention $\hbar = u = 1$. It is easy to reintroduce \hbar and u by counting the dimensions. The square of a vector 2+1-vector is given by $q^2 \equiv q_0^2 - u^2 \mathbf{q}^2 \equiv q_0^2 - \mathbf{q}^2$. The one-loop self-energy $\Sigma_1(q)$ is given by

$$\Sigma_1(q) = \Lambda_0 \int \frac{d^3 k}{(2\pi)^3 i} \frac{1}{(q - k)^2 + i\delta} \frac{1}{k^2 - M_0^2 + i\delta}. \quad (72)$$

M_0 is the bare mass of the Higgs boson, and is the same quantity as that denoted as $M_{h'}$ in the previous section. The change in the notation is so as to be more clear when discussing the mass renormalization. In the same way, let us call the bare coupling Γ_0 . This is the same quantity as that denoted as Γ in the previous subsection. Λ_0 in the above equation is given by

$$\Lambda_0 = -\Gamma_0^2. \quad (73)$$

This is the bare coupling, to which we shall calculate the corrections which arise due to the vertex correction diagrams (and not due to the vacuum polarization diagrams). We have included the factor 2 for the two Goldstone modes. As explained below eqn. (68), the factor 2 for identical Higgs bosons has already been taken into account. The choice of the sign is such that positive Λ_0 corresponds to repulsion and negative Λ_0 to attraction. Λ_0 has the dimension $(\text{energy})^5 \times \text{area}$.

Let us say that \mathbf{q} is small compared with the zone boundaries, so that we will not need to worry about the boundary effects. In this regard, u is small compared with the electron velocity, but it is not small here since we are not comparing it against the electron velocity. Since $\hbar u \pi / a = \Delta E$, we do not have to worry about boundary effects so long as the Goldstone boson does not carry energy that is comparable with ΔE .

Eqn. (72) can be integrated easily using standard methods⁹. First, the usual procedure of combining the denominators yields

$$\Sigma_1(q) = \Lambda_0 \int_0^1 d\lambda \int \frac{d^3\ell}{(2\pi)^3 i} [\ell^2 + \lambda (q^2(1 - \lambda) - M_0^2) + i\delta]^{-2}. \quad (74)$$

The Wick rotation yields

$$\Sigma_1(q) = \Lambda_0 \int_0^1 d\lambda \int \frac{d^3\tilde{\ell}}{(2\pi)^3} [-\tilde{\ell}^2 + \lambda (q^2(1 - \lambda) - M_0^2) + i\delta]^{-2}. \quad (75)$$

After evaluating this Euclidean integral, we obtain

$$\Sigma_1(q) = \frac{\Lambda_0}{8\pi} \int_0^1 d\lambda [\lambda (M_0^2 - q^2(1 - \lambda)) - i\delta]^{-1/2}. \quad (76)$$

Finally, under the condition $M_0^2 > q^2 > 0$, we obtain

$$\Sigma_1(q) = \frac{\Lambda_0}{4\pi |q|} \tanh^{-1} \frac{|q|}{M_0}, \quad (77)$$

where $|q| = \sqrt{q^2}$. Near $|q| = M_0$, this diverges logarithmically, reflecting the long-distance confinement. In the region $|q| < M_0$, Σ has the following expansion:

$$\Sigma_1 \approx \frac{\Lambda_0}{4\pi M_0^3} \left(M_0^2 + \frac{q^2}{3} + \frac{q^4}{5M_0^2} + \dots \right). \quad (78)$$

Let us retain the first two terms. This approximation is accurate to 10% up to $|q|/M_0 = 0.7$ or so.

Since we have the following general relation:

$$G^{-1} = Z^{-1}(q^2 - M^2) = q^2 - M_0^2 - \Sigma, \quad (79)$$

we can immediately write down the one-loop Z and M^2 as

$$Z_1^{-1} = 1 - \frac{\Lambda_0}{12\pi M_0^3}, \quad \frac{M_1^2}{M_0^2} = Z_1 \left(1 + \frac{\Lambda_0}{4\pi M_0^3} \right). \quad (80)$$

This corresponds to a (short-distance) correction to the Higgs-boson mass of the form

$$M_1^2 = M_0^2 - \Sigma_1 \approx M_0^2 + \frac{\Lambda_0}{4\pi M_0} = M_0^2(1 + x). \quad (81)$$

The dimensionless coupling x is defined in eqn. (69).

For the sake of completeness, it is relatively easy to extend the preceding analysis to the case with an artificial small mass m for the (pseudo-)Goldstone boson, which screens

the interaction with a characteristic distance \hbar/mu . In this case, the one-loop self-energy correction is given by

$$\Sigma_1 = \frac{\Lambda_0}{4\pi|q|} \tanh^{-1} \frac{|q|}{M_0 + m} \quad (82)$$

We again see that the self-energy diverges near the threshold, corresponding to the long-distance correction mediated by the Goldstone boson which goes almost on shell.

Let us now consider the vertex correction. The corresponding Feynman diagram is shown in fig. 4.

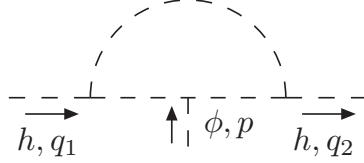


FIG. 4. The one-loop vertex-correction diagram.

The one-loop amplitude is given by

$$\Gamma_1/\Gamma_0 - 1 = \Lambda_0 \int \frac{d^3k}{(2\pi)^3 i} \frac{1}{k^2 + i\delta} \frac{1}{(q_1 - k)^2 - M_0^2 + i\delta} \frac{1}{(q_2 - k)^2 - M_0^2 + i\delta}. \quad (83)$$

Γ_1 is Γ_0 plus the the one-loop correction. Proceeding in a similar fashion to that for the self-energy, we obtain the following Feynman parameter integral:

$$\Gamma_1/\Gamma_0 - 1 = \frac{\Lambda_0}{16\pi} \int_0^1 d\lambda_1 d\lambda_2 d\lambda_3 \delta(1 - \sum_{i=1}^3 \lambda_i) \left[(\lambda_1 M_0^2 - q_1^2 \lambda_2 \lambda_3) + (\lambda_2 M_0^2 - q_2^2 \lambda_1 \lambda_3) - p^2 \lambda_1 \lambda_2 - i\delta \right]^{-3/2}. \quad (84)$$

In the general case, eqn. (84) is difficult to handle analytically, even though the first two integrations are relatively easy. We have been able to evaluate it with the aid of the on-line software¹⁰ based on MATHEMATICA, but the result is cumbersome and so we do not reproduce it here.

Here we are only interested in the constant term which arises from eqn. (84). This corresponds to the limit in which the external momenta are small compared with M_0 . In particular, when p is small, the vertex-correction amplitude is related to the self-energy amplitude by

$$\Gamma_1/\Gamma_0 - 1 = \frac{\partial \Sigma_1}{\partial M_0^2}. \quad (85)$$

This can be deduced from the form of eqn. (72). We then take $q^2 \ll M_0$. The application of eqn. (85) to the leading term of eqn. (78) yields

$$\Gamma_1/\Gamma_0 - 1 \approx -\frac{\Lambda_0}{8\pi M_0^3}. \quad (86)$$

From eqn. (80), the one-loop estimate for the critical coupling is $-\Lambda_0 \approx 4\pi M_0^3$, at which point the one-loop mass becomes zero, and above which the mass becomes tachyonic. At this value of $-\Lambda_0$, by eqn. (86), the one-loop vertex correction makes the effective vertex greater by 50%. Clearly, finite-order calculations are insufficient to deal with such corrections and so a non-perturbative method is needed. In the next subsection, we shall sum such contributions to all orders using the Dyson–Schwinger equations.

Before proceeding, we note that a tachyonic Higgs mass by itself only suggests the instability of the AF ground state, and not the form of the true ground state. For instance, one may very well conclude at this stage that the true ground state is given simply by $v = 0$, i.e., zero magnetization, which would be a reasonable interpretation for tachyonic Higgs mass. We shall show, however, that this is only partially true. In the all-order analysis, the real part of the Higgs-mass squared does indeed go negative above a certain value of the coupling, but ground-state instability occurs regardless of the sign of the real part of the Higgs-mass squared.

C. Dyson–Schwinger equations

We have shown that finite-order treatment is insufficient. In particular, we would like to sum the finite short-distance contributions to all orders. Let us deal with this problem by making use of the Dyson–Schwinger equations.

First, the self-energy diagram is shown in fig. 5. This includes all possible perturbative corrections that involve the $hh\phi$ coupling, other than vacuum polarization which we either discard, or assume that it is taken into account in the Goldstone boson propagator. The non-perturbative equation is thus a modification of eqn. (78), given by

$$\Sigma = (q^2 - M_0^2) - Z^{-1}(q^2 - M^2) = \frac{\Gamma\Gamma_0}{4\pi M^3} \left(M^2 + \frac{q^2}{3} + \dots \right). \quad (87)$$

M , Σ and $\Gamma = \sqrt{-\Lambda}$ are now all-order quantities. Eqn. (87) sums only the finite terms in the higher-order corrections, and not the singular infra-red contributions.

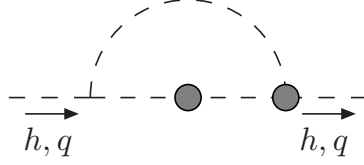


FIG. 5. The self-energy diagram corresponding to the Dyson-Schwinger equation. The blobs represent all-order corrections.

Let us introduce the following notation:

$$y = \frac{M}{M_0}, \quad w = \frac{\Sigma(q=0)}{M_0^2}, \quad g = \frac{\Gamma}{\Gamma_0}. \quad (88)$$

y and w are different from the quantities that appeared in sec. III B. y is the normalized mass, which we would like to obtain as a function of the normalized coupling x defined by eqn. (69). w is the normalized self-energy and g gives the coupling renormalization. From eqn. (87), we derive the following:

$$w = y^2 Z^{-1} - 1 = \frac{3}{4}(y^2 - 1), \quad xyg = w(w + 1). \quad (89)$$

As for the vertex correction, we can write it as the sum of two diagrams, as shown in fig. 6.

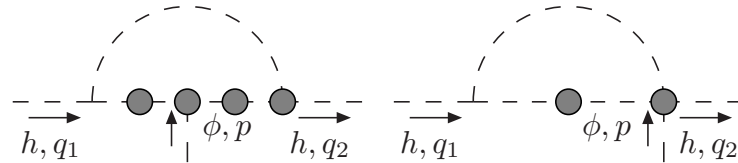


FIG. 6. The two diagrams whose sum contains all of the vertex-correction contributions.

The second of these diagrams contains a 4-point vertex, and it would ordinarily be necessary to construct another Dyson-Schwinger equation to describe it. However, as we saw in sec. IV B, so long as we are only interested in the constant part of the corrections, the insertion of a Goldstone-boson vertex is formally equivalent to the derivative by M_0^2 and multiplication by Γ_0 . That is, the sum of the two diagrams in fig. 6 is given simply by

$$\Gamma/\Gamma_0 - 1 = \frac{\partial \Sigma}{\partial M_0^2} \quad (90)$$

The partial derivative means differentiating by M_0^2 while keeping Λ_0 constant. Now Σ is a function of both Λ_0 and M_0^2 , but by counting the dimensions, we see that Σ must be of the form

$$\Sigma = M_0^2 w(x). \quad (91)$$

Thus eqn. (90) can be written as

$$g = 1 + w - \frac{3x}{2} \frac{dw}{dx}. \quad (92)$$

To proceed, we make use of eqn. (89) to eliminate g from eqn. (92). This yields the following ordinary, first-order and non-linear differential equation:

$$\frac{3x}{2} \frac{dw}{dx} = (1 + w) \left(1 - \frac{w}{xy} \right). \quad (93)$$

In order to obtain $w = (3/4)(y^2 - 1)$ as a function of x , we then need to solve eqn. (93) with the boundary condition

$$w(0) = 0. \quad (94)$$

Unfortunately, this boundary condition makes the solution of eqn. (93) difficult. We do not think that eqn. (93) can be integrated analytically and, on the other hand, numerical implementation suffers from instability near $x = 0$ where eqn. (93) contains $0/0$.

Let us adopt a power-series solution. Simply expanding y in powers of x and substituting in eqn. (93), we are able to read off the coefficients with the boundary condition given by eqn. (94). We used MAXIMA¹¹ for preliminary algebra and the evaluation of the first 28 coefficients, and FORM¹² after that. Using FORM, we calculated the first 102 coefficients, which, in our implementation, is equivalent to having verified the first 101. On a LINUX machine equipped with a 2.6 GHz Pentium 4 processor, the evaluation of the last coefficient took just over 5 minutes to perform. The machine has 2 GB memory, but less than 1 % of the memory was taken up by the calculation. The first few coefficients are given by

$$y = 1 + \frac{2}{3}x - \frac{7}{9}x^2 + 2x^3 - \frac{641}{81}x^4 + \frac{2321}{54}x^5 - \mathcal{O}(x^6). \quad (95)$$

It should be noted that x being the normalized coupling, the expansion in x should match with the finite terms from the perturbative calculation, order by order. For example, the truncation of eqn. (95) at the first order in x reproduces the one-loop perturbative result, eqn. (80), at the first power order in Λ_0 .

This series is divergent for all non-zero x . In other words, the critical coupling x_c , which is calculated as

$$x_c = \lim_{n \rightarrow \infty} \frac{a_{n-1}}{a_n}, \quad (96)$$

vanishes. Here, a_n stands for the coefficient of the n 'th power of x . The radius of convergence of the series is given by $|x_c|$.

We plot the ratio appearing in eqn. (96), for finite n , with n up to 10 in fig. 7. We have found that these are numerically consistent with

$$\frac{a_{n-1}}{a_n} = \left[\frac{11}{6} - \frac{3n}{2} + \mathcal{O}(n^{-1}) \right]^{-1}. \quad (97)$$

The coefficient of the $\mathcal{O}(n^{-1})$ contribution is about 0.3, numerically. We see that for large n , the series is factorially divergent, and the radius of convergence is therefore zero.

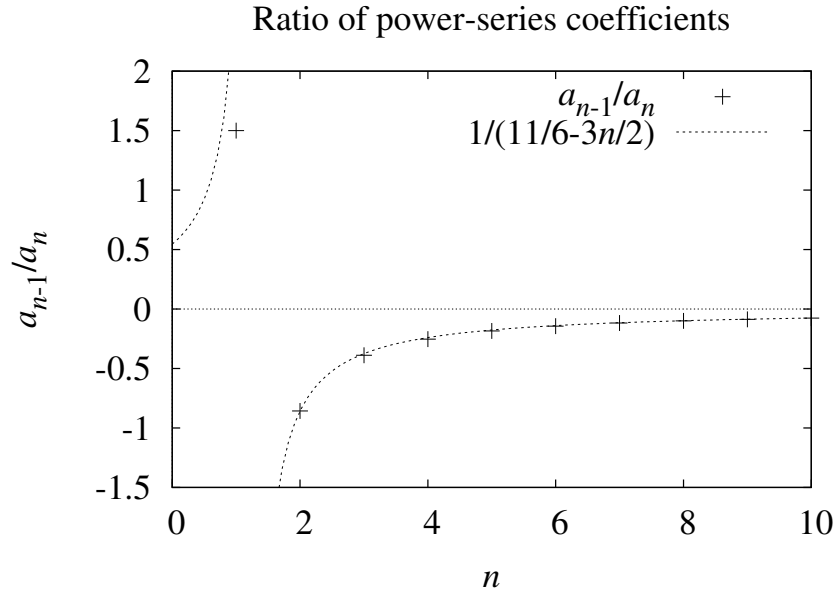


FIG. 7. The ratio a_{n-1}/a_n , for $1 \leq n \leq 10$, compared with the expression $(11/6 - 3n/2)^{-1}$.

D. Borel summation

A method which is commonly used to assign a meaning to factorially divergent expansions is that of Borel summation. For $Y(X)$ given by the power series

$$Y(X) = \sum_{n=0}^{\infty} A_n X^{-n}, \quad (98)$$

the Borel transformed series is given by

$$\mathcal{B}Y(t) = \sum_{n=0}^{\infty} \frac{A_{n+1}}{n!} t^n. \quad (99)$$

$Y(X)$ is then recovered by the Laplace transform:

$$Y(X) = A_0 + \int_0^{\infty} \mathcal{B}Y(t) e^{-Xt} dt. \quad (100)$$

Since eqn. (100) requires $X > 0$, let us define

$$X = 1/x, \quad A_n = a_n, \quad y(x) = Y(1/x), \quad (101)$$

when $x > 0$, and

$$X = -1/x, \quad A_n = (-1)^{n+1} a_n, \quad y(x) = -Y(-1/x), \quad (102)$$

when $x < 0$. We can then readily find the Borel transformed series $\mathcal{B}Y(t)$.

Alternatively, using the first two terms of eqn. (97), we can estimate the Borel sum to be

$$\begin{cases} \mathcal{B}Y(t) \approx a_1(1 + 3t/2)^{-7/9} & (x > 0), \\ \mathcal{B}Y(t) \approx a_1(1 - 3t/2)^{-7/9} & (x < 0). \end{cases} \quad (103)$$

In fig. 8, we compare this estimation, with $a_1 = 2/3$, against the series expansion using the actual values of a_n up to a_{100} . For the range of $\mathcal{B}Y(t)$ shown in fig. 8, the contribution of the last term in the expansion is at most about 0.5% of the total number, and therefore we consider the error due to the truncation of the power series to be negligible. The two expressions match reasonably well. We shall henceforth make use of the estimation of eqns. (103) in our discussions.

The first of eqns. (103), which corresponds to positive x , has a unique Laplace transform. In other words, the series is Borel summable. The result of the Laplace transform is

$$y(x) = a_0 + a_1(2/3)^{7/9} e^{2/3x} x^{2/9} \Gamma(2/9, 2/3x) \quad (x > 0), \quad (104)$$

again using the approximation of taking the first two terms of eqn. (97). Γ is the upper incomplete Gamma function, defined by

$$\Gamma(s, a) = \int_a^{\infty} t^{s-1} e^{-t} dt. \quad (105)$$

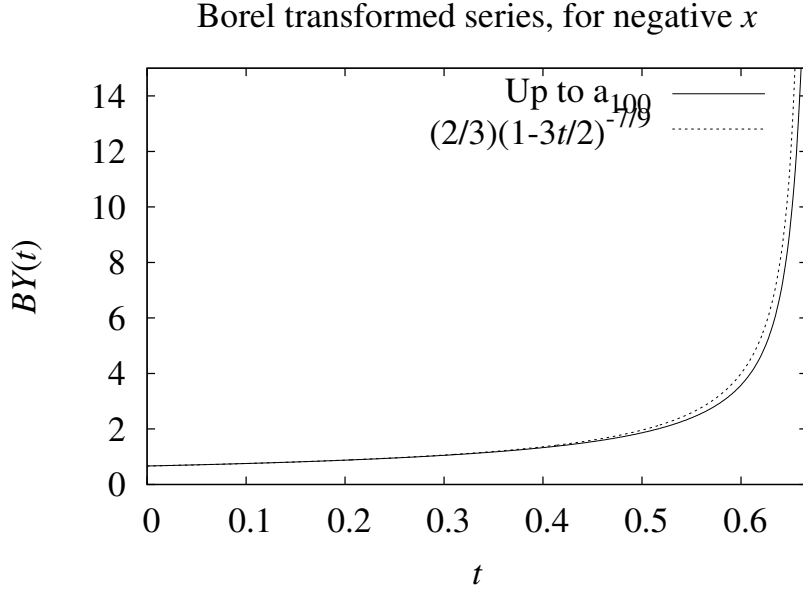


FIG. 8. $\mathcal{B}Y(t)$ for negative x , using the actual values of a_n (solid line, with $n \leq 27$), and the estimation of eqns. (103) (dotted line).

On the other hand, the second of eqns. (103), which corresponds to negative x , has a branch-point singularity at $t = 2/3$. In other words, the series is non-Borel summable. We can still derive the expression for $y(x)$ formally and, not too surprisingly, the result turns out to be of the same form as eqn. (104), but is dependent on the choice of the contour of integration. Let us choose the contour such that we obtain a negative imaginary part for the self-energy, corresponding to the retarded Green's function. This choice of the contour is obtained by moving the singularity downwards in the complex t plane, as shown in fig. 9.

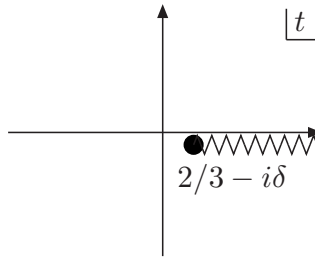


FIG. 9. Moving the branch cut off the real axis. The contour of integration is along the positive real axis.

This is equivalent to giving a negative and infinitesimal imaginary part to x , and so in

the end, we have, for both positive and negative x , as well as $x = 0$,

$$y(x) = 1 + \frac{4}{9} \exp\left(\frac{2/3}{x - i\delta}\right) \left(\frac{2/3}{x - i\delta}\right)^{-2/9} \Gamma\left(\frac{2}{9}, \frac{2/3}{x - i\delta}\right). \quad (106)$$

The definition of the incomplete Gamma function, as given by eqn. (105), is ambiguous for negative or complex a . A more general formula reads

$$\Gamma(s, z) = \Gamma(s) - \gamma(s, z) = \Gamma(s) - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{z^{s+k}}{s+k}, \quad (107)$$

using which we are able to evaluate $y(x)$ numerically. $\gamma(s, z)$ is the lower incomplete Gamma function. The resulting form of $y(x)$, and $y^2(x)$, is shown in fig. 10.

In fig. 10, we also show the function

$$- (4/9) \Gamma(2/9) \sin(2\pi/9) e^{2/3x} (-2/3x)^{-2/9}, \quad (108)$$

for negative x . This is an $x \rightarrow -\infty$ limit of the imaginary part of eqn. (106), and it agrees well with the imaginary part of $y(x)$. In general, we expect that any singularity at $t = t_0$ in the Borel transform induces a term of the form $e^{t_0/x}$. There is nothing new in this divergent behaviour of the perturbation series, the so-called renormalon¹³ being a celebrated example. As for the exponential form of the Borel sum, since the imaginary part of the Green's function corresponds to the binding energy of the condensate to which the AF vacuum decays, this implies a BCS-like behaviour of the energy gap, as given by eqn. (62).

We note that, unlike in finite-order calculations, the criticality, in the sense of an imaginary part of y , develops already at $x = -0$, even though it is quite small until $-x$ becomes comparable with $2/3$.

The real part of y remains positive even at large $-x$. This is against our naive expectation that strong attractive interaction makes the single-particle mass negative. The behaviour seen here suggests that there is a balance between the attractive interaction mediated by the Goldstone boson and the contribution due to the condensate which counteracts it. The former is dominant at small $-x$, and the latter is dominant at large $-x$.

The real part of y^2 becomes negative at $x \approx -0.9$. Naively speaking, AF order co-exists with the new condensate state for $x \gtrsim -0.9$, and is destroyed below this value of x .

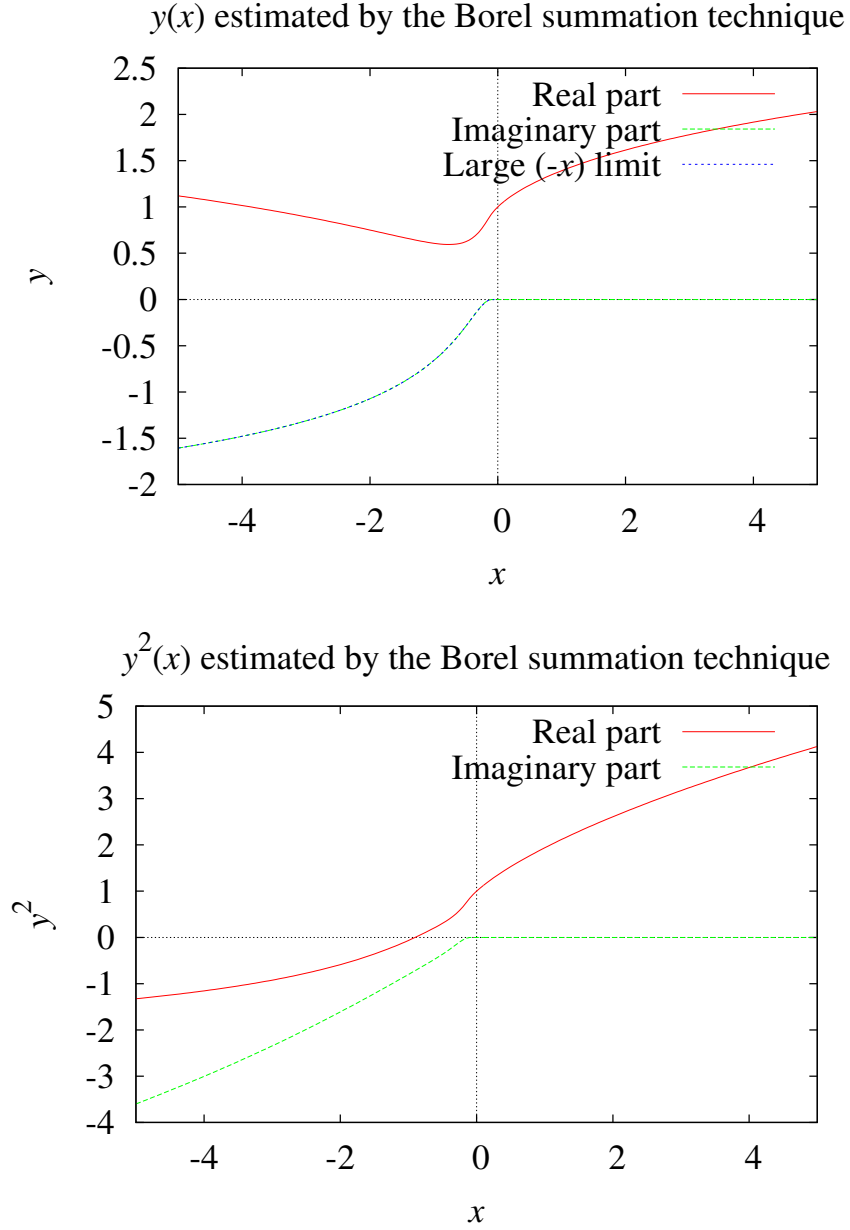


FIG. 10. $y(x)$ (upper figure), and $y^2(x)$ (lower figure), evaluated as the Laplace transform of eqn. (103).

E. Phenomenological discussions

In the preceding sections, we were able to deduce the non-perturbative behaviour of the single-particle Green's function for the Higgs boson.

As per the conventional wisdom⁸, the imaginary part of the self-energy determines the relaxation time of the unstable ground state which, in turn, and by the uncertainty principle,

is the inverse of the binding energy. The energy gap is thus given simply by

$$2\Delta = -\text{Im } M \equiv -M_0 u^2 \text{Im } y. \quad (109)$$

The total condensation energy, per unit area, is given by the two-dimensional phase-space integral

$$\frac{E_{\text{cond}}}{A} = - \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \text{Im} \sqrt{\mathbf{k}^2 + M^2} = -\frac{1}{6\pi} \text{Im} ((K^2 + M^2)^{3/2} - M^3). \quad (110)$$

K stands for the upper limit of integration, which is close to π/a . In the limit as the K tends to infinity, we obtain

$$\frac{E_{\text{cond}}}{A} = -\frac{1}{4\pi} K \text{Im } M^2. \quad (111)$$

This expression, for very large $-x$, has the behaviour $(-x)^{4/9}$, or proportional to Δ^2 .

We do not know how to calculate T_C , since long-range order is known to disappear for finite temperature in two spatial dimensions.

Let us define S_0 by

$$-2/3x = S/S_0. \quad (112)$$

By eqn. (70), we have

$$S_0 = \frac{3\pi}{32} \sin^2 \theta (1 - 3 \cos^2 \theta)^2, \quad (113)$$

and so S_0 is $\mathcal{O}(10^{-2})$ to $\mathcal{O}(10^{-1})$.

Using eqn. (108) and then using eqn. (23), the superconducting gap is estimated as

$$2\Delta_{\text{SC}} = 1.17\Delta E e^{-S/S_0} (S/S_0)^{-2/9} = \frac{1.17S_0}{g(\mu)a^2} e^{-S/S_0} (S/S_0)^{7/9}. \quad (114)$$

This is shown in fig. 11. For large S , the exponential factor tends to suppress superconductivity, whereas for small S , the ΔE factor suppresses it. The maximum of $2\Delta_{\text{SC}}$ as a function of S/S_0 occurs at $S/S_0 = 7/9$.

Small S and large S correspond to over-doping and under-doping, respectively, and therefore the general behaviour is consistent with the doping behaviour of high- T_C superconductors, assuming that T_C is proportional to Δ . Out of the parameters in eqns. (113, 114), S_0 can in principle be measured through the measurement of the magnetization at the peak of $2\Delta_{\text{SC}}$, but one needs to be careful here since S refers to the magnetization when superconductivity is not present. Perhaps some extrapolation of the magnetization at higher temperatures can be made to the magnetization at 0 K. This, together with the measurement of $g(\mu)$ and a , gives us $2\Delta_{\text{SC}}$ as a function of S , to be compared against experiment.

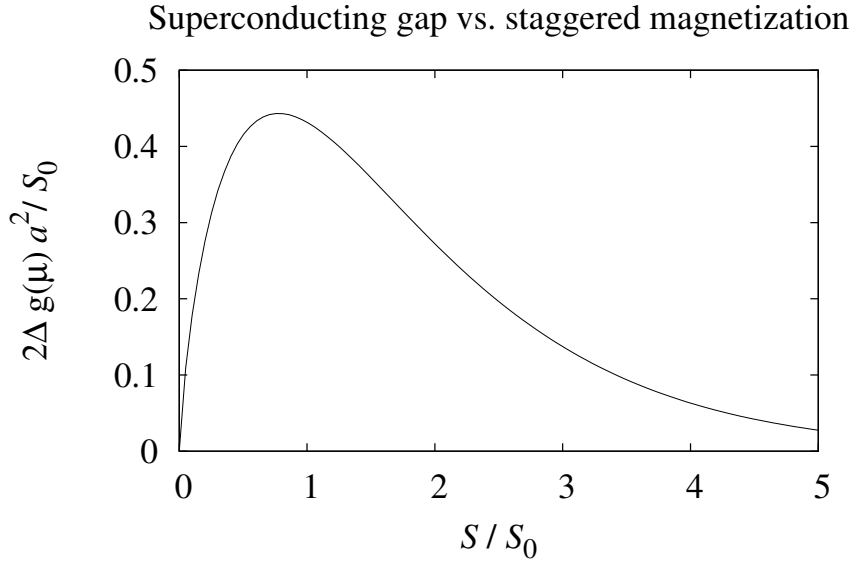


FIG. 11. The superconducting gap, normalized to $c_S J^2 a^2 g_F$, as a function of the mean staggered magnetization S normalized to S_0 .

In the sense of the positivity of the real part of the renormalized $M^2 = M_0^2 y^2$, superconductivity co-exists with AF down to about $x = -0.90$, or $S/S_0 = 0.74$ ($0.74 < 7/9 = 0.77\ldots$). AF disappears in the over-doping region. The pseudo-gap is naturally associated with the excitation of e to e_* , which is characterized by ΔE .

As for the overall magnitude of Δ_{SC} , an $\mathcal{O}(100 \text{ meV})$ superconducting gap is obtained when $1/a^2 g(\mu)$ is $\mathcal{O}(1 \sim 10 \text{ eV})$. This seems quite reasonable. The superconducting gap goes up when the electron density goes down (against naive expectation). That is, smaller values of $1/m_{\text{eff}}$ is favoured for the effective electron mass.

As for the symmetry of the superconducting gap, our analysis does not single out a particular type of symmetry, but we expect, naively, that the requirement of forming spin-singlet pairs in AF background restricts pair formation in the $(\pm\pi, \pm\pi)$ directions, and so there are necessarily nodes in these directions.

V. CONCLUSIONS

We made an adaptation of Gribov's axial-current conservation techniques to the situation of spin-current conservation in systems with partial magnetic order. We obtained the form

factor, or the coupling strength of the Goldstone bosons, and the mass of the Higgs boson in a non-perturbative manner. We also obtained relationships among the parameters of the theory. Our analysis is of a general nature and rely only on conventional wisdom relating to quasiparticle dynamics in the momentum space, but many of the results were obtained using the approximation that ΔE , the spin excitation energy, is independent of energy or momentum. Some other results depend on the assumption that the inverse of the electron Green's function is linear in energy. These assumptions are reasonable when the spin-wave velocity is small.

Our approach is distinct from methods that start from Hamiltonians with localized states as the bases, because the Pauli exclusion principle acting on localized states is an ingredient of the latter formulations. The Pauli exclusion principle is present in our approach without such restrictions, in the form of the Feynman propagators, i.e., in the form of the analytical behaviour of the Green's functions.

We then discussed two applications of our framework.

We first discussed the case of fast magnons, i.e., near the Mott insulating state, with linear dispersion relation in three spatial dimensions. This case admits a treatment using the Gribov equation, which sums the leading overlapping divergences to all orders. The solution of the Gribov equations leads to a peculiar solution in which the fermionic degrees of freedom seem to disappear, and we are left with something which can be interpreted as a tower of bosonic excitations. A natural interpretation for this phenomenon is as a string-like one-dimensional excitation, which may correspond to the path transgressed by a wrong-spin electron. This phenomenon occurs when the spin-wave velocity is large, which implies large magnetization and small carrier density, i.e., near the Mott insulator state.

With increased carrier density, the system starts to lose the magnetic order, and the spin-wave velocity starts to fall. When the spin-wave velocity falls below the electron velocity, i.e., $u < v_F$, the behaviour of the system changes qualitatively, because the collective response of the vacuum subsides and the electrons become more free, even though magnetization still remains. This cross-over occurs for reasonably large values of S . We have found that $u < v_F$ is satisfied unless $g(\mu)$ is small, v_F is small and S is large, i.e., quite near the Mott insulator state. This finding supports our being able to use the nearly-free electron picture to discuss high- T_C superconductivity.

When discussing high- T_C superconductivity, we proposed that there is a mixing between

the Goldstone and Higgs modes due to Goldstone-boson condensation. Goldstone-boson condensation is stabilized by superconductivity, and superconductivity cannot arise without Goldstone-boson condensation. This gives rise to an effective Higgs–Higgs–Goldstone three-point interaction.

We then discussed the stability of the AF ground state with respect to the collapse due to this three-point interaction, which is attractive. In view of the observation of high- T_C superconductivity in two-dimensional AF systems, we focused on the two-dimensional case, which is easier in the sense that the short-distance integral converges.

The exchange of Goldstone magnons induces non-analyticity in the Higgs-boson Green's function. This implies that the AF ground state is unstable at low temperature. The analysis of the Higgs-boson Green's function was carried out in an all-order manner, by using a Dyson–Schwinger summation of the relevant finite, short-distance, contribution to all orders. We found a solution for the mass renormalization as a power series in the coupling constant. The power series is factorially divergent, but can be treated using the Borel summation method.

We found that an essential singularity develops in the mass renormalization when the coupling constant is zero. We thus conclude that the true AF ground state in two spatial dimensions contains a condensate into which the AF state decays. The energy gap associated with this condensate has the behaviour that is expected for superconductivity, and we argued that it indeed is the superconducting pair condensate. Superconductivity co-exists with AF in the under-doped region.

A number of results of this paper can be tested experimentally. First, the expression for the superconducting energy gap. Second, the presence of the Higgs boson at larger values of carrier concentration, and the prediction of its mass. Third, the tower of bosonic states in AF systems at smaller values of carrier concentration, especially in the case of three spatial dimensions. In addition, the analytical relationships between such quantities as ΔE , $g(\mu)$, S and u are all testable.

What might be the signals of Goldstone-boson condensation? We can at the moment only think of indirect signatures such as the lowering of the Higgs-boson excitation energy from ΔE , which may be too delicate to be treated as a concrete signal.

Large T_C can be achieved by making $g(\mu)$ small, or by reducing the effective mass of the electrons, without destroying the magnetic interaction.

ACKNOWLEDGMENTS

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